Master's Thesis

Continuous Bernoulli Distribution and p-adic Periodic Zeta Function

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Abstract

The p-adic Dirichlet L-function or the Kubota-Leopoldt p-adic L-function is a p-adic version of Dirichlet L-function. It is defined by the *Mellin-Mazur transform* of the Bernoulli distribution μ_k on $\mathbb{Z}_p^\times$ for an integer $k \geq 1$. In this thesis, we extend the Bernoulli distribution with integer parameters to a continuous version i.e., a Bernoulli number with a continuous parameter. Also, we generalize the p-adic Dirichlet zeta functions that is defined only for Dirichlet characters to the p-adic periodic zeta functions that is defined for arbitrary periodic functions with values in $\mathbb{C}_p$.

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1 Introduction

The p-adic Dirichlet L-function or the Kubota-Leopoldt p-adic L-function is a p-adic version of Dirichlet L-function. In other words, it is a p-adic analytic function on $\mathbb{Z}_p$ that interpolates the special values of Dirichlet L-functions at the non-positive integers. It possesses several properties that the complex Dirichlet L-functions enjoy.

The importance of p-adic Dirichlet L-functions comes from several roles played by the function in the Iwasawa theory. For example, the vanishing of μ-invariant of the p-adic Dirichlet L-function conjectured by Iwasawa in 1960 and had been an open problem until Ferrero-Washington [1] finally resolved in 1979. Relevant problems and generalizations can be found in [2], [3], [6], [7], [4], [14], [15], [16], and [17].

The p-adic Dirichlet L-function is defined by the *Mellin-Mazur transform* of the Bernoulli distribution μ_k on $\mathbb{Z}_p^\times$ for an integer $k \geq 1$ (See [5], [8], [9], and [18] for other constructions). One can show that the Bernoulli distribution satisfies an additive property on the disjoint union of compact-open subsets of $\mathbb{Z}_p$, which comes from the properties of the Bernoulli polynomial. Also, using Mazur's treatment, one can normalize the Bernoulli distribution in order to get a p-adically bounded distribution, i.e., a p-adic measure, say $\mu_{k,\alpha}$ for a p-adic integer α. A famous identity present a relation between the measures $\mu_{1,\alpha}$ and $\mu_{k,\alpha}$.

The Mellin-Mazur transform of a measure is the integration of the continuous character $x \mapsto (x/\omega(x))^s\chi(x)$ for the Teichmüller character ω and a Dirichlet character χ on $\mathbb{Z}_p^\times$, with respect to the measure. This amounts to the complex version, namely that the Dirichlet L-function is a Mellin transform of the continuous character $y \mapsto y^s$ with respect to the measure $e^{-2\pi y}\frac{dy}{y}$ on the group $\mathbb{R}_{>0}^\times$.

In this thesis, we will generalize the previous discussion to the case of p-adic zeta functions for arbitrary periodic functions with values in $\mathbb{C}_p$ (see section 6). For this generalization, we will first extend the Bernoulli distribution with integer parameters to a continuous version, i.e., a Bernoulli number with a continuous parameter (see section 5). By a Bernoulli number with a continuous parameter, we mean a p-adic analytic functions that interpolate modified Bernoulli numbers.

Following the previous construction in a similar way, we also define a distribution $\mu_{s,\lambda}$ on $\mathbb{Z}_p$ for a continuous parameter $s \in \mathbb{Z}_p$ and a periodic function λ (see Definition 5.1). We also adopt the treatment of Mazur in order to obtain another distribution $\mu_{s,\lambda,\kappa,\alpha}$ that depends on p-adic integers s, α, and two periodic functions λ and κ by normalizing. In fact, we present a precise condition on the periodic functions λ and κ in order for $\mu_{s,\lambda,\kappa,\alpha}$ to be p-adically bounded (see Definition 5.2 and Theorem 5.4 (1)).

By extending the previous result on the relation between $\mu_{1,\alpha}$ and $\mu_{k,\alpha}$, we obtain a relation between $\mu_{s,\lambda,\kappa,\alpha}$ and $\mu_{1,\lambda,\kappa,\alpha}$. In particular, it can be shown that a limit $\lim_{s\to 0} s^{-1}d\mu_{s,\lambda,\kappa,\alpha}$ of measures, is again a measure (see Theorem 5.4 (2)) and it also can be written in terms of $x/\omega(x)$ and $\mu_{1,\lambda,\kappa,\alpha}$. This result is not visible when one considers only the Bernoulli distribution with the integer parameters. In addition, the same discussion applies to distributions B_s and D_s defined by p-adic analytic functions (see Definition 5.5).

2 Preliminaries

This section describes the materials that will be necessary in the latter part of thesis. The main reference is [12]. Please refer to the reference for more explanation.

2.1 Bernoulli number

In this section, we present basic definitions on the Bernoulli number that will be used in latter part of thesis.

Definition 2.1. *Define the Bernoulli numbers, denoted* B_n *as the* n*-th coefficient of a power series*

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!}.$$

From the series, one can get $B_{2k+1} = 0$ for $k \geq 1$. Also, it is already known that

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ \ldots$$

The following definition will play an important role in this thesis.

Definition 2.2. *Define the* n*-th Bernoulli polynomial as*

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

Since

$$e^{xt} = \sum_{i=0}^{\infty} \frac{(xt)^i}{i!},$$

one can easily get a generating function as follows,

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k(x) t^k}{k!}. \tag{1}$$

2.2 p-adic numbers

Let us review basic definitions of p-adic numbers. For this, we review first on a general theory on the completion of a metric space.

Definition 2.3. *Let* $\|\cdot\|$ *be a norm on a field* F*. Define a metric on field* F *by setting the distance between* x *and* y *as* $d(x, y) = \|x - y\|$.

Definition 2.4. *A metric space is a set F with a metric d that assigns a nonnegative real number $d(x, y)$ for every $x, y \in F$ satisfying the following:*

(1) $d(x, y) \geq 0$ *and* $d(x, y) = 0 \Longleftrightarrow x = y$*,*

(2) $d(x, y) = d(x, y)$*,*

(3) $d(x, y) + d(y, z) \geq d(x, z)$*. (The triangle inequality)*

Definition 2.5. *Let F be a field and let d be a metric on F. A completion of a metric space (F, d) is a pair consisting of a complete metric space (F^*, d^*) and an isometry $\phi : F \to F^*$ such that $\phi[F]$ is dense in F^*.*

Let F be a field with norm $\|\cdot\|$. Also, let R be the set of all Cauchy sequences $\{a_n\}_{n=1}^{\infty}$ with respect to the norm $\|\cdot\|$ when $a_n \in F$. Define addition and multiplication of sequences as follows:

$$\{a_n\}_{n=1}^{\infty} + \{b_n\}_{n=1}^{\infty} = \{a_n + b_n\}_{n=1}^{\infty}$$
$$\{a_n\}_{n=1}^{\infty} \times \{b_n\}_{n=1}^{\infty} = \{a_n b_n\}_{n=1}^{\infty}.$$

Then it is a tedious task to show that $(R, +, \times)$ is a commutative ring. A *null Cauchy sequence* is defined as a sequence that converges to zero. Let $\mathfrak{m}$ be the set of null Cauchy sequences. Then the set $\mathfrak{m}$ is a subset of R and it can be easily verified that it is a maximal ideal of R. Thus $R/\mathfrak{m}$ is a field.

The field F can be embedded in R with the map $a \mapsto (a, a, ...)$ and obviously $(a, a, ...)$ is a Cauchy sequence. Then the field F can be considered as a subfield of $R/\mathfrak{m}$. Thus $R/\mathfrak{m}$ is the completion of F with respect to $\|\cdot\|$.

From now on, we will see a specific metric space $\mathbb{Q}$ with the following norm:

Definition 2.6. *Let p be a prime number and x a rational number $x \neq 0$. Express x as $x = p^{v_p(x)}y$ where y is a rational number coprime to p and $v_p(x)$ is an integer. Define the p-adic absolute value of x, denoted $|x|_p$ as*

$$|x|_p := p^{-v_p(x)}.$$

When $x = 0$, set $|0|_p = 0$.

One can show with no difficulty that the distance function $d(x, y) = |x - y|_p$ satisfies three properties in Definition 2.4. In particular, it can be easily checked that the triangle inequality comes from the following inequality:

$$|x + y|_p \leq max(|x|_p, |y|_p). \tag{2}$$

Furthermore, the equality in (2) holds when $|x|_p \neq |y|_p$.

Let $F = \mathbb{Q}$ with the usual absolute value. Then by above construction, one can get the field of real numbers. Now, consider that $F = \mathbb{Q}$ with p-adic norm $|\cdot|_p$. Then by completing $\mathbb{Q}$ with respect to the new distance, one can also get a new field that is called the field of *p-adic numbers*, denoted by $\mathbb{Q}_p$.

Definition 2.7. *Define the set of p-adic integers as follows:*

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

Remark 2.1.

1. $\mathbb{Z}_p$ is a subring of $\mathbb{Q}_p$.

2. It can be shown that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$.

3. One can write elements of $\mathbb{Z}_p$ as formal power series as follows:

$$\sum_{n=0}^{\infty} b_n p^n, 0 \leq b_n \leq p-1.$$

Now, consider the function $f(x) = a^x$ when a is a fixed number. Consider in the case of complex analysis, one can easily compute the derivative of $f(x)$ i.e., $f'(x) = (\log a)a^x$. In the p-adic case, although the proof is not easy, one can get a similar result. Then by this result, one can get the following formula,

$$\log_p a = \lim_{n\to\infty} \frac{a^{p^n} - 1}{p^n}.$$

This formula can be interpreted as the p-adic derivative of a^x at $x = 0$. So, the conclusion is that $(a^{p^n} - 1)/p^n$ behave like the logarithm.

We will introduce here the N-th partial sum of the p-adic expansion of α denoted by $(\alpha)_N$. Suppose that $\alpha \in \mathbb{Z}_p$. Let $(\alpha)_N$ be the rational integer that congruent to $\alpha (\bmod p^N)$. If

$$\alpha = a_0 + a_1 p + a_2 p^2 + \cdots,$$

then

$$(\alpha)_N = a_0 + a_1 p + \cdots + a_{n-1} p^{N-1}.$$

2.3 Hensel's lemma

We introduce well-known Hensel's lemma here to explain the Teichmüller character in Section 2.4.

Theorem 2.1 (Hensel's lemma)**.** *Let $f(x) \in \mathbb{Z}_p[x]$ be a polynomial that coefficients are in $\mathbb{Z}_p$. Suppose that $f(x_1) \equiv 0 \pmod p$ and $f'(x_1) \not\equiv 0 \pmod p$. Then there exists a unique p-adic integer y such that $f(y) = 0$ and $y \equiv x_1 (\bmod p)$.*

Proof. We prove inductively that if

$$f(x) \equiv 0 \pmod{p^n}$$

has a solution a_n then it can be lifted uniquely to $a_{n+1} (\bmod p^{n+1})$ such that

$$f(a_{n+1}) \equiv 0 \pmod{p^{n+1}}$$

and $a_{n+1} \equiv a_n (\bmod\, p^n)$. Indeed, if we write $a_{n+1} = p^n t + a_n$, then we get

$$f(p^n t + a_n) \equiv 0 \ (\bmod\, p^{n+1}).$$

And write $f(x) = \sum_i c_i x^i$, then we obtain

$$\begin{aligned} f(p^n t + a_n) &= \sum_i c_i (a_n + p^n t)^i \\ &\equiv \sum_i c_i (a_n^i + i p^n t a_n^{i-1}) \ (\bmod\, p^{n+1}) \\ &\equiv f(a_n) + p^n t f'(a_n) \ (\bmod\, p^{n+1}). \end{aligned}$$

Now, we need to solve for t in the congruence

$$p^n t f'(a_n) + f(a_n) \equiv 0 \ (\bmod\, p^{n+1}).$$

Since $f(a_n) \equiv 0 (\bmod\, p^n)$ this reduces to

$$t f'(a_n) \equiv -(f(a_n)/p^n) \ (\bmod\, p)$$

which has a unique solution t $(\bmod\, p)$ because $a_n \equiv a_1$ $(\bmod\, p)$ and $f'(a_1) \not\equiv 0$ $(\bmod\, p)$. Also, the sequence $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence, and limit of this sequence is the required solution. Since a_{n+1} is a unique lifting $(\bmod\ p^{n+1})$ of a_n $(\bmod\, p^n)$, the uniqueness of the limit solution is clear. This completes the proof of the theorem. □

2.4 Teichüller character

Now let the polynomial $f(x) = x^m - 1$ with $m | p-1$. We already know that $(\mathbb{Z}/p\mathbb{Z})^{\times}$ is a cyclic group of order $p-1$. So $f(x) \equiv 0 (\bmod\, p)$ has m distinct solutions. Consider the case when $p > 2$ and $m = p-1$. By Hensel's lemma, for each $1 \leq i \leq p-1$, there exists a number $\omega(i) \in \mathbb{Z}_p$ such that $\omega(i) \equiv i (\bmod\, p)$ and $\omega(i)^{p-1} = 1$. These $\omega(i)$ are called the Teichmüller representatives of the residue classes mod p. Now, consider the map

$$(\mathbb{Z}/p\mathbb{Z})^{\times} \to \mathbb{Z}_p$$

given by $x \mapsto \omega(x)$. Then this map defines the Teichmüller character. And the Teichmüller character is a multiplicative character. $\omega(x)$ is defined by $\omega(x \text{ mod } p)$ for $x \in \mathbb{Z}_p^{\times}$. For every $x \in \mathbb{Z}_p^{\times}$, one can write x uniquely as $x = \omega(x)\langle x \rangle$ when $\langle x \rangle \in 1 + p\mathbb{Z}_p$. For $p \neq 2$, we have

$$\omega(x) = \lim_{n \to \infty} x^{p^n},$$

(By using induction, it is easy to verify that $a \equiv b$ (mod p) implies $a^{p^n} \equiv b^{p^n}$ (mod p^{n+1}), so that)

$$\omega(x) = \omega(x)^{p^n} = \frac{x^{p^n}}{\langle x \rangle^{p^n}}$$

and the denominator will go to 1 when $n \to \infty$.

3 p-adic integration

Note that in the p-adic topology, all sets (the author of [12] calls them "intervals") of the form

$$a + p^N \mathbb{Z}_p$$

are both closed and open. The Riemann integral can be built over these sets. Suppose that X is a compact-open subset of $\mathbb{Q}_p$ (such as an "interval", $\mathbb{Z}_p$ or $\mathbb{Z}_p^\times$). Then a *p-adic distribution* μ on X can be regarded as a $\mathbb{Q}_p$-linear functional on the $\mathbb{Q}_p$-vector space of locally constant functions on X. Recall that a function f is locally constant if for every $x \in X$ there exists a neighbourhood U such that f is constant in U. Let c be a constant and χ_U be the characteristic function of U. Then any locally constant function can be expressed in a linear combination of functions of the form $c\chi_U$. Now, one can write $\mu(U)$ for $\mu(\chi_U)$. Let us present another definition:

Definition 3.1. *A p-adic distribution μ on X is an additive map from the set of compact-open sets in X to $\mathbb{Q}_p$. That is, if $U \subseteq X$ is a disjoint union of compact-open sets $U_1, U_2, ..., U_n$, then*

$$\mu(U) = \mu(U_1) + \cdots + \mu(U_n).$$

A typical "interval" in $\mathbb{Q}_p$ is of the form

$$a + p^n \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x - a|_p \leq p^{-n}\}.$$

A basis of open sets for $\mathbb{Q}_p$ is the collection of such sets. So, a p-adic distribution is determined by its values on such "intervals". This idea is explained in the following theorem, which is the fundamental property of distributions on $\mathbb{Q}_p$.

Theorem 3.1. *Every map μ from the set of "intervals" of X to $\mathbb{Q}_p$ satisfying*

$$\mu(a + p^n \mathbb{Z}_p) = \sum_{b=0}^{p-1} \mu(a + bp^n + p^{n+1}\mathbb{Z}_p) \tag{3}$$

when $a + p^n \mathbb{Z}_p \subseteq X$, extends uniquely to a p-adic distribution on X.

Proof. Every compact-open sets $U \subseteq X$ can be expressed as a finite disjoint union of intervals, i.e., $U = \bigcup_i I_i$. So, one can write

$$\mu(U) = \sum_i \mu(I_i).$$

Now, one have to check that well-definedness of this. Assume that $U = \bigcup_i I_i$ and $U = \bigcup_i J_i$, when I_i and J_i are "intervals". Then one can refine these decompositions to $\bigcup_{ij} I_{ij}$ when the I_{ij} are intervals. Thus the result is verified and uniqueness is also easy to verified. □

In the p-adic integration, the locally constant function is used like the step function of the Riemann integration. By above theorem, $\int_x f d\mu$ can be defined using locally constant functions.

First, when U_i are compact-open sets, one can write f as a linear combination of characteristic functions χ_{U_i}, i.e.,

$$f = \sum_i c_i \chi_{U_i}.$$

Then $\int_x f d\mu$ can be defined as follows,

$$\int_X f d\mu = \sum_i c_i \mu(U_i).$$

Example 3.1. Recall that for each positive integer k, $B_k(x)$ is the k-th Bernoulli polynomial. The Bernoulli distribution is defined as

$$\mu_k(a + p^n \mathbb{Z}_p) = p^{n(k-1)} B_k(a/p^n).$$

It is necessary to check that (3) holds, i.e., it is necessary to show that

$$p^{n(k-1)} B_k(a/p^n) = \sum_{b=0}^{p-1} p^{(n+1)(k-1)} B_k((a + bp^n)/p^{n+1})$$

holds. Now multiply $p^{-n(k-1)}$ on both sides of this equation. Then the identity to be proved follows from

$$B_k(px) = p^{k-1} \sum_{b=0}^{p-1} B_k(x + \frac{b}{p}) \tag{4}$$

which is easily deduced from the power series generating function for the Bernoulli polynomials as follows. Recall the equation (1):

$$\sum_{k=0}^{\infty} \frac{B_k(x) t^k}{k!} = \frac{te^{xt}}{e^t - 1}.$$

Then one can get

$$\sum_{k=0}^{\infty} B_k(px) \frac{t^k}{k!} = \frac{te^{pxt}}{e^t - 1}.$$

On the other hand, the right hand side of the equation (4) is the coefficient of t^k in the power series expansion of

$$p^{k-1} \sum_{b=0}^{p-1} \frac{te^{(x+b/p)t}}{e^t - 1} = \frac{p^{k-1} te^{xt}}{e^t - 1} \sum_{b=0}^{p-1} e^{bt/p} = \frac{p^{k-1} te^{xt}}{e^t - 1} \cdot \frac{e^t - 1}{e^{t/p} - 1} = \frac{p^k (t/p) e^{px(t/p)}}{e^{(t/p)} - 1},$$

and the coefficient of t^k on the right hand side is $B_k(px)$. So one can conclude that μ_k is a p-adic distribution.

Now, we introduce here a p-adic measure to define p-adic integration.

Definition 3.2. *A distribution μ of X is a p-adic measure on X if*

$$|\mu(U)|_p \leq B$$

where B is a constant and for all compact-open $U \subseteq X$.

We are ready to give the p-adic version of the Riemann integration:

Theorem 3.2. *Let μ be a p-adic measure on X and let $f : X \to \mathbb{Q}_p$ be a continous function. Then the "Riemann sums"*

$$S_N := \sum_{\substack{0 \le a < p^N \\ a+p^N\mathbb{Z}_p \subseteq X}} f(x_{a,N})\mu(a + p^N\mathbb{Z}_p)$$

(where the sum is taken over all a for which $a + p^N\mathbb{Z}_p \subseteq X$ and $x_{a,N}$ is chosen in the "interval" $a + p^N\mathbb{Z}_p$) converge to a limit in $\mathbb{Q}_p$ as $N \to \infty$. This limit is independent of the $\{x_{a,N}\}$.

Proof. First, it is necessary to check that the sequence $\{S_N\}$ is Cauchy sequence. Let X be a finite union of intervals. Choose large enough N so that every interval $a + p^N\mathbb{Z}_p$ is contained in X or disjoint from X. Suppose that $M > N$. Let $\tilde{a}$ be the least non-negative residue of a mod p^N. Then, since μ holds additivity, S_N can be rewritten as

$$S_N = \sum_{\substack{0 \le a < p^M \\ a+p^M\mathbb{Z}_p \subseteq X}} f(x_{\tilde{a},N})\mu(a + p^M\mathbb{Z}_p).$$

Let $\epsilon > 0$. Since X is compact and f is a continuous function, f is uniformly continuous. Then for large enough N, $|f(x) - f(y)|_p \le \epsilon$ when $x \equiv y (\bmod p^N)$. Thus, we have

$$\begin{aligned} |S_N - S_M|_p &= \Big| \sum_{\substack{0 \le a < p^M \\ a+p^M\mathbb{Z}_p \subseteq X}} (f(x_{\tilde{a},N}) - f(x_{a,M}))\mu(a + p^M\mathbb{Z}_p) \Big|_p \\ &\le \max_a \Big(|f(x_{\tilde{a},N}) - f(x_{a,M})|_p \cdot |\mu(a + p^M\mathbb{Z}_p|_p \Big) \\ &\le B\epsilon, \end{aligned}$$

since $x_{\tilde{a},N} \equiv x_{a,M} \pmod{p^N}$. So, one can conclude that the "Riemann sums" have a limit. Also it is easy to see that the limit does not depend on the choice of the $\{x_{a,N}\}$. The theorem is verified. ☐

Assume f is a function that satisfies the hypothesis of the above theorem. Then denote the limit of the Riemann sums by the symbol

$$\int_X f d\mu.$$

Now, choose $x_{a,N} = a$. Then the above integral is equivalent to the following limit

$$\lim_{N \to \infty} \sum_{0 \le a < p^N} f(a)\mu(a + p^N\mathbb{Z}_p).$$

Suppose that μ is a distribution and $\alpha \in \mathbb{Q}_p$. Then $\alpha\mu$ is again a distribution. Now, let $\alpha \in \mathbb{Z}_p^\times$, and we define μ' as $\mu'(U) = \mu(\alpha U)$. Then also μ' is again a distribution.

4 p-adic interpolation

Most of materials in this section comes from [12].

Given a sequence of integers $\{a_k\}_{k=1}^{\infty}$, one can ask when we get a continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ such that $f(k) = a_k$. Since $\mathbb{Z}_p$ is compact, such a continuous function is uniformly continuous and bounded. Thus, the following condition is a necessary condition: for each m, there exist an integer $N = N(m)$ such that

$$k \equiv k' (\operatorname{mod} p^N) \Rightarrow a_k \equiv a_{k'} (\operatorname{mod} p^m). \tag{5}$$

That is, if k and k' are close p-adically, then a_k and $a_{k'}$ are also close p-adically. Conversely, whenever (5) is hold, and

$$\sup_n |a_n|_p$$

is bounded, then one can get a continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$, and the function is as follows. Let n_i be a sequence of integers tending to x for $x \in \mathbb{Z}_p$. Now define

$$f(x) = \lim_{i\to\infty} f(n_i).$$

The limit exists by (5). In addition, if a_i' is any other sequence tending to x, one can easily see that

$$\lim_{i\to\infty} f(n_i) = \lim_{i\to\infty} f(n_i')$$

so that the function is well-defined. Therefore, the interpolation problem can be reduced to checking (5). These observations can be stated as follows:

Theorem 4.1. *Assume that a sequence of numbers $f(k)$, for $k = 0, 1, 2, ...$ is p-adically bounded and satisfies the following condition. For each natural number m, there exist a natural number $N = N(m)$ such that*

$$k \equiv k' (\operatorname{mod} p^N) \Rightarrow f(k) \equiv f(k') (\operatorname{mod} p^m).$$

Then, there exist a continuous function $\tilde{f} : \mathbb{Z}_p \to \mathbb{Z}_p$ such that $\tilde{f}(k) = f(k)$.

As an application of the above theorem, notice that if $n \equiv 1 (\operatorname{mod} p)$, the function $f(k) = n^k$ can be p-adically interpolated since the condition (5) is easily verified. In fact, $f(s) = n^s$ is a p-adically analytic function for $s \in \mathbb{C}_p$ satisfying $|s|_p < p^{\frac{p-2}{p-1}}$. For odd primes p, this region is larger than the unit disc.

Recall that the classical theorem of Weierstrass: a continuous function on a closed interval can be uniformly approximated by polynomials. The p-adic version of this theorem is Mahler's theorem. This theorem says that for any continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$, there are numbers c_k in $\mathbb{Q}_p$ satisfying

$$f(x) = \sum_{k=0}^{\infty} \binom{x}{k} c_k.$$

Therefore the sequence of partial sums of this series gives the polynomial approximations for $f(x)$.

Now, begin with some observations from combinatorial analysis. The proof of Mahler's theorem will follow a treatment due to Bojanic.

Suppose that the p-adic expansions of n and k are as follows:

$$\begin{aligned} n &= a_0 + a_1 p + a_2 p^2 + \cdots \\ k &= b_0 + b_1 p + b_2 p^2 + \cdots . \end{aligned}$$

Then it is easy to see that for prime p,

$$\binom{n}{k} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\binom{a_2}{b_2}\cdots \pmod p.$$

Especially,

$$\binom{p^n}{k} \equiv 0 (\bmod p)$$

for $1 \leq k \leq p^n - 1$.

Recall the binomial inversion formula:

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \tag{6}$$

if and only if

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k. \tag{7}$$

Now fix a positive integer m, and let

$$a_k = \begin{cases} (-1)^k & \text{if } k = m \\ 0 & \text{otherwise.} \end{cases}$$

Then the above binomial inversion formula gives the following relation:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{k}{m} = \begin{cases} (-1)^m & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Suppose that $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is continuous, and let

$$a_n(f) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k). \tag{9}$$

Then one can get

$$f(n) = \sum_{k=0}^{n} \binom{n}{k} a_k(f)$$

by (6). This suggests how to solve the interpolation problem. Given a sequence $f(0), f(1), f(2), \ldots$, define $a_n(f)$ by (9) and let

$$f(x) := \sum_{k=0}^{\infty} \binom{x}{k} a_k(f). \tag{10}$$

If one can prove that this is a p-adically convergent series, it is done. Observe that

$$\binom{x}{n} = \begin{cases} x(x-1)\cdots(x-n+1)/n! & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}$$

is a continuous function of a p-adic variable and takes integer values for $x \in \mathbb{Z}$. So, by continuity,

$$\left|\binom{x}{n}\right|_p \leq 1$$

for all $x \in \mathbb{Z}_p$. The main idea in Mahler's theorem is to show that $|a_k(f)|_p \to 0$ as $k \to \infty$ when (5) is holds.

Now, new operators Δ^n is defined as follows,

$$\Delta^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k).$$

So we get

$$\begin{aligned} &\Delta f(x) = f(x+1) - f(x), \\ &\Delta^2 f(x) = f(x+2) - 2f(x+1) + f(x), \\ &\cdots. \end{aligned}$$

And it is clear that Δ^n is a linear operator. The following lemmas will be used:

Lemma 4.2.

$$\Delta^n f(x) = \sum_{j=0}^{m} \binom{m}{j} \Delta^{n+j} f(x-m).$$

Proof. To prove this lemma, it is enough to show that

$$f(x) = \sum_{j=0}^{m} \binom{m}{j} \Delta^j f(x-m).$$

Then the result can be derived by applying Δ^n to both sides of the above equation. But

$$\begin{aligned} \sum_{j=0}^{m} \binom{m}{j} \Delta^j f(x-m) &= \sum_{j=0}^{m} \binom{m}{j} \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} f(x-m+k) \\ &= \sum_{k=0}^{m} (-1)^k f(x-m+k) \sum_{j=0}^{m} \binom{m}{j} \binom{j}{k} (-1)^j. \end{aligned}$$

By (8), the inner sum is zero except for $k = m$. Where $k = m$, it is $(-1)^m$. The result is now clear. $\square$

Lemma 4.3. *For any sequence $f(0), f(1), f(2), f(3), \ldots,$ define*

$$a_n(f) := \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k).$$

Then,

$$\sum_{j=0}^{m} \binom{m}{j} a_{n+j}(f) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k+m).$$

Proof. If $m = 0$, the result is clear. By Lemma 4.2,

$$\Delta^n f(m) = \sum_{j=0}^{m} \binom{m}{j} \Delta^{n+j} f(0).$$

But

$$\Delta^n f(m) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(m+k)$$

and

$$\Delta^n f(0) = a_n(f).$$

Then the result is now immediate. $\square$

Theorem 4.4 (Mahler, 1961)**.** *Assume that $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is continuous function. Let*

$$a_n(f) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k).$$

Then $a_n(f) \to 0$ as $n \to \infty$. Thus the series

$$\sum_{k=0}^{\infty} \binom{x}{k} a_k(f)$$

converges uniformly in $\mathbb{Z}_p$. Moreover,

$$f(x) = \sum_{k=0}^{\infty} \binom{x}{k} a_k(f).$$

Proof. Since $\mathbb{Z}_p$ is compact and f is continuous, it is uniformly continuous. So, for any positive integer s, there is a positive integer t such that

$$|x - y|_p \leq p^{-t} \Rightarrow |f(x) - f(y)|_p \leq p^{-s}, \text{ where } x, y \in \mathbb{Z}_p.$$

In particular, for $k = 0, 1, 2, \ldots$,

$$|f(k + p^t) - f(k)|_p \leq p^{-s}.$$

Since f is continuous on $\mathbb{Z}_p$ and $\mathbb{Z}_p$ is compact, it is bounded there. Without loss of generality, let $|f(x)|_p \leq 1$ for all $x \in \mathbb{Z}_p$. Then, $|a_n(f)|_p \leq 1$ for all n. By Lemma 4.3, and the formula for a_n given there, one can get

$$a_{n+p^t}(f) = -\sum_{j=1}^{p^t - 1} \binom{p^t}{j} a_{n+j}(f) + \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \{f(k + p^t) - f(k)\}.$$

In first sum of the above formula, p divides each of the binomial coefficients. So, one can get the following inequality:

$$\begin{aligned} |a_{n+p^t}(f)|_p &\leq \max_{1 \leq j < p^t} \{p^{-1} |a_{n+j}(f)|_p, p^{-s}\} \\ &\leq p^{-1} \text{ for } n \geq p^t \end{aligned}$$

since $|a_n(f)|_p \leq 1$. Now if n is replaced by $n + p^t$ in the penultimate inequality, one can get the following inequality:

$$|a_n(f)|_p \leq p^{-2} \text{ for } n \geq 2p^t.$$

And if one repeat the argument $(r-1)$ times, one can get

$$|a_n(f)|_p \leq p^{-s} \text{ for } n \geq sp^t.$$

So, by above argument, $a_n(f) \to 0$ as $n \to \infty$. Therefore, the series

$$\sum_{k=0}^{\infty} \binom{x}{k} a_k(f)$$

converges to $f(x)$ since both are continuous and equal on a dense set and

$$\left| \binom{x}{k} \right|_p \leq 1$$

for $x \in \mathbb{Z}_p$. □

Remark 4.1. This is the p-adic analogue of the classical theorem of Weierstrass that any continuous function on $[-1, 1]$ can be uniformly approximated by polynomials.

Corollary 4.5. *Let $\{a_n\}$ be any sequence in $\mathbb{C}_p$. Define*

$$b_n := \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k.$$

Then there exists a continuous function $f : \mathbb{Z}_p \to \mathbb{C}_p$ such that $f(n) = a_n$ if and only if $b_n \to 0$ as $n \to \infty$.

Proof. This is clear from Mahler's theorem. □

Now consider the convergence problem for $x \in \mathbb{C}_p$. When certain conditions holds, one can prove that the above series extends to an analytic function for $|x|_p < R$ where $R > 1$. To achieve such a result, one can follow Washington's approach.

Lemma 4.6. *Let $P_i(x) = \sum_{n=0}^{\infty} a_{n,i} x^n$ be a sequence of power series that converge in some fixed subset D of $\mathbb{C}_p$. Assume that for each n, $a_{n,i} \to a_{n,0}$ as $i \to \infty$. And suppose that for each $x \in D$ and $\epsilon > 0$ there exist an $n_0 = n_0(x, \epsilon)$ such that*

$$\left| \sum_{n \geq n_0} a_{n,i} x^n \right| < \epsilon$$

uniformly in i. Then $\lim_{i \to \infty} P_i(x) = P_0(x)$ for all $x \in D$.

Proof. Let $\epsilon > 0$ and $x \in D$ be given. And choose n_0 as the above. Then for sufficiently large i, one can get

$$|P_0(x) - P_i(x)| \leq \max_{n < n_0} \{\epsilon, |a_{n,0} - a_{n,i}|_p |x|_p^n\} = \epsilon.$$

□

Theorem 4.7. *Suppose $r < p^{-1/(p-1)} < 1$ and*

$$f(x) = \sum_{n=0}^{\infty} \binom{x}{n} a_n$$

for $x \in \mathbb{C}_p$ when $|a_n|_p \leq Mr^n$ for some M. Then, $f(x)$ can be written as a power series. And the radius of convergence of that power series is at least $R = (rp^{1/(p-1)})^{-1}$.

Proof. First, consider the partial sums

$$P_i(x) = \sum_{n \leq i} \binom{x}{n} a_n = \sum_{n \leq i} a_{n,i} x^n.$$

Then

$$a_{n,i} = a_n \frac{\text{integer}}{n!} + a_{n+1} \frac{\text{integer}}{(n+1)!} + \cdots.$$

Since $\mathrm{ord}_p j! \leq j/(p-1)$, one can get

$$|a_{n,i}|_p \leq \max_{j \geq n} |a_j/j!\,|_p \leq \max_{j \geq n} M r^j p^{j/(p-1)} \leq MR^{-n}.$$

And also

$$\begin{aligned} |a_{n,i} - a_{n,i+k}|_p &= \left| a_{i+1} \frac{\text{integer}}{(i+1)!} + \cdots + a_{i+k} \frac{\text{integer}}{(i+k)!} \right|_p \\ &\leq MR^{-(i+1)} \to 0 \end{aligned}$$

as $i \to \infty$. So, by the above, one can conclude that $\{a_{n,i}\}_{i=1}^{\infty}$ is a Cauchy sequence. Let $a_{n,0} = \lim_{i \to \infty} a_{n,i}$. Then $|a_{n,0}|_p \leq MR^{-n}$. Now let

$$P_0(x) = \sum_{n=0}^{\infty} a_{n,0} x^n.$$

Then for $x \in \mathbb{C}_p$, P_0 converges where $|x|_p < R$. Since the P_i are polynomials, they converge in the following region,

$$D = \{x \in \mathbb{C}_p : |x|_p < R\}.$$

Furthermore,

$$\left| \sum_{n \geq n_0} a_{n,i} x^n \right| \leq \max_{n \geq n_0} MR^{-n} |x|_p^n \to 0$$

as $n_0 \to \infty$ uniformly in i for $|x|_p < R$. Therefore for $|x|_p < R$, $|P_0(x) - P_i(x)|_p \leq \epsilon$ uniformly. So, the result is now clear. □

Remark 4.2. Since the above R is bigger than 1, i.e., $R > 1$, the series in the above theorem has a wider radius of convergence.

By Mahler's theorem, when (5) is hold,

$$f(x) = \sum_{n=0}^{\infty} \binom{x}{n} a_f(n)$$

is a continuous function on $\mathbb{Z}_p$. Also, suppose that the growth condition of Theorem 4.7 holds in the sequence $a_n = f(n)$. Then one can get a p-adic analytic continuation of $f(x)$ with a wider domain. As we show below in Theorem 4.8, one can sharpen the Theorem 4.7.

Theorem 4.8. *Let $f : \mathbb{Z}_p \to \mathbb{C}_p$ be a continuous function with the Mahler series*

$$f(x) = \sum_{k=0}^{\infty} \binom{x}{k} c_k.$$

Then, f is the restriction of an analytic power series

$$\sum_{n=0}^{\infty} a_n x^n$$

with $a_n \to 0$ if and only if $|c_k/k!\,|_p \to 0$ as $k \to \infty$.

Proof. The natural basis of the vector space of polynomials:

$$1, x, x^2, \ldots.$$

But one can also use the basis

$$1, (x)_1, (x)_2, \ldots,$$

where $(x)_k = x(x-1)\ldots(x-k+1)$. Let $s(n,k)$ be the Stirling numbers of the first kind. Then one can get the formulas:

$$(x)_n = \sum_{k=0}^{n} s(n,k) x^k.$$

Also, let $S(n,k)$ be the Stirling numbers of the second kind. Then one can get

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k.$$

The Stirling numbers of the second kind interpret combinatorically the number of partitions of a set of n elements into k blocks. In other words, $S(n,k)$ interpret the number of equivalence relations on a set of n elements with k equivalence classes. Therefore, if one express $f(x) = \sum_n a_n x^n$, one can also express it as $\sum_k b_k (x)_k$. And it is already known that the Stirling numbers are integers. So, one can get

$$\sup_n |a_n|_p = \sup_k |b_k|_p.$$

But $c_k = k!\, b_k$. So the result follows immediately. $\square$

Suppose that $n \equiv 1 (\operatorname{mod} p)$. Then one can get

$$n^s = (1 + (n-1))^s = \sum_{k=0}^{\infty} \binom{s}{k} (n-1)^k.$$

And we can see that if we apply Theorem 4.7, then it gives us a p-adic continuation of n^s.

5 Continuous Bernoulli Distribution and p-adic periodic zeta function

The modified Bernoulli numbers $Q^{k-1}\omega^{-k}(a)B_k(a/Q)$, $k \geq 0$ for $p|Q, 1 \leq a < Q$ can be interpolated by an analytic function on $\mathbb{Z}_p$. Indeed the analytic function $B(s,a,Q)$ in $s \in \mathbb{Z}_p$ defined by

$$B(s,a,Q) := \frac{\langle a\rangle^s}{Q}\sum_{n=0}^{\infty}\binom{s}{n}\left(\frac{Q}{a}\right)^n B_n \tag{11}$$

interpolates the numbers

$$B(k,a,Q) = \frac{\langle a\rangle^s}{Q}\sum_{n=0}^{\infty}\binom{s}{n}\left(\frac{Q}{a}\right)^n B_n = Q^{k-1}\omega(a)^{-k}B_k\left(\frac{a}{Q}\right).$$

Definition 5.1. *Regarding λ as a periodic function with the period Np^m, for $s \in \mathbb{Z}_p$ we define a function $\mu_{s,\lambda}$ on basic open subsets of $\mathbb{Z}_p^\times$ such that*

$$\mu_{s,\lambda}(a + p^m\mathbb{Z}_p) = \sum_{\substack{r\equiv a(p^m)\\ 0\leq r<Np^m}} \lambda(r)B(s,r,Np^m). \tag{12}$$

Proposition 5.1. *Let $s = k \in \mathbb{Z}_{\geq 1}$, and $\lambda = \omega^k$. Then $\mu_{k,\omega^k} = \mu_k$ is the k-th Bernoulli distribution.*

Proof. We omit this proof. □

Proposition 5.2. *Let $\lambda_k(n) = \xi^n\omega^k(n)$ for an N-th root of unity ξ, then*

$$\mu_{k,\lambda_k}(a + p^m\mathbb{Z}_p) = \left(T\frac{d}{dT}\right)^{k-1}\frac{T^{(a)_m}}{T^{p^m-1}-1}\Big|_{T=\xi}.$$

In particular, $\mu_{1,\lambda_1}(a + p^m\mathbb{Z}_p) = \frac{\xi^{(a)_m}}{\xi^{p^m}-1}$ is a measure on $\mathbb{Z}_p$, which is considered in [9] to deduce several properties of p-adic Dirichlet L-function.

Proof. By the above definitions, we have

$$\mu_{k,\lambda_k}(a + p^m\mathbb{Z}_p) = (p^mN)^{k-1}\sum_{\substack{r\equiv a(p^m)\\ 0\leq r<Np^m-1}} \xi^r B_k\left(\frac{r}{Np^m}\right). \tag{13}$$

Since $B_k(x) = \left(\frac{d}{dt}\right)^k \frac{te^{tx}}{e^t-1}\Big|_{t=0}$, the above formula (13) is equal to

$$\begin{aligned}
(Np^m)^{k-1}\sum_{q=0}^{N-1}\xi^{p^mq+a}B_k\Big(\frac{p^mq+a}{Np^m}\Big) &= (Np^m)^{k-1}\sum_q \xi^{p^mq+a}\Big(\frac{d}{dt}\Big)^k \frac{te^{(\frac{p^mq+a}{Np^m})t}}{e^t-1}\Big|_{t=0} \\
&= (Np^m)^{k-1}\xi^a\Big(\frac{d}{dt}\Big)^k \frac{t\sum\limits_{q=0}^{N-1}\xi^{p^mq}e^{(\frac{p^mq+a}{Np^m})t}}{e^t-1} \\
&= (Np^m)^{k-1}\xi^a\Big(\frac{d}{dt}\Big)^{k-1} \frac{te^{\frac{a}{p^mN}t}\Big(\sum\limits_{q=0}^{N-1}\xi^{p^mq}e^{\frac{qt}{N}}\Big)}{e^t-1} \\
&= (Np^m)^{k-1}\xi^a\Big(\frac{d}{dt}\Big)^{k-1} \frac{e^{\frac{a}{p^mN}t}\Big(\sum\limits_{q=0}^{N-1}\xi^{p^mq}e^{\frac{qt}{N}}\Big)}{e^t-1}\Big|_{t=0}. \qquad (14)
\end{aligned}$$

Let $T = e^{\frac{1}{p^mN}t}$. Then we have $\frac{d}{dt} = \frac{1}{p^mN}\cdot T\frac{d}{dT}$, and the formula (14) is equal to

$$\begin{aligned}
(Np^m)^{k-1}\xi^a\Big(\frac{1}{p^mN}T\frac{d}{dT}\Big)^{k-1}&\frac{T^a\Big(\sum\limits_{k=0}^{N-1}\xi^{p^mq}T^{p^mq}\Big)}{T^{p^mN}-1}\Big|_{T=1} \\
&= \Big(T\frac{d}{dT}\Big)^{k-1}\frac{(\xi T)^a\frac{(\xi T)^{p^mN}-1}{T^{p^m}\xi^{p^m}-1}}{T^{p^mN}-1} \\
&= \Big(T\frac{d}{dT}\Big)^{k-1}\frac{(\xi T)^a}{(\xi T)^{p^m-1}}\Big|_{T=1}. \qquad (15)
\end{aligned}$$

Now let $S = \xi T$. Then the above formula (15) is equal to $\Big(S\frac{d}{dS}\Big)^{k-1}\frac{S^a}{S^{p^m}-1}\Big|_{S=\xi}$. Thus we conclude that

$$\mu_{k,\lambda_k}(a+p^m\mathbb{Z}_p) = \Big(T\frac{d}{dT}\Big)^{k-1}\frac{T^{(a)_m}}{T^{p^m}-1}\Big|_{T=\xi}$$

holds.

Now we will show that $\mu_{1,\lambda_1}(a+p^m\mathbb{Z}_p)$ is a measure on $\mathbb{Z}_p$. Since $\xi^{p^m-1}\neq 0$, we have $\xi^{p^m}\in\{\xi,\xi^2,\xi^3,\dots,\xi^{N-1}\}$. So, $|\xi^{p^m}-1|_p\in\{|\xi-1|_p,|\xi^2-1|_p,\dots,|\xi^{N-1}-1|_p\}$. Now let $B = \min\limits_{1\le i\le N-1}(|\xi^i-1|_p) > 0$. Then we have $\dfrac{1}{|\xi^{p^m}-1|_p}\le B$. And since $|\xi^{(a)_m}|_p = 1$, we get $\Big|\dfrac{\xi^{(a)_m}}{\xi^{p^m}-1}\Big|_p\le B$. Therefore, we conclude that $\mu_{1,\lambda_1}(a+p^m\mathbb{Z}_p)$ is a measure on $\mathbb{Z}_p$. □

For general λ, we have

Proposition 5.3. *The function $\mu_{s,\lambda}$ becomes a distribution on $\mathbb{Z}_p^\times$.*

Proof. Let us consider the sum

$$\sum_{q=0}^{p-1}\mu_{s,\lambda}(a+p^mq+p^{m+1}\mathbb{Z}_p) = \sum_{q=0}^{p-1}\sum_{\substack{r=a+p^mq(p^m+1)\\ 1\le r<Np^{m+1}}}\lambda(r)B(s,r,Np^{m+1}). \qquad (16)$$

Setting $r = Np^m l + t$ when $0 \le l < p$ and $0 \le t < Np^m$. Since $r \equiv a + p^m q \pmod{p^{m+1}}$, we have $Np^m l + t \equiv a + p^m q \pmod{p^{m+1}}$. Then we obtain that $(Nl - q)p^m \equiv a - t \pmod{p^{m+1}}$. Now let $t \equiv a \pmod{p^m}$. Then we get $q \equiv Nl - \frac{a-t}{p^m} \pmod{p}$. Hence the above formula (16) is equal to

$$\sum_{t} \lambda(t) \sum_{l=0}^{p-1} B(S, Np^m l + t, Np^{m+1}). \tag{17}$$

Now, we will check that

$$\sum_{l=0}^{p-1} B(s, Np^m l + t, Np^{m+1}) = B(s, t, Np^m) \text{ with } p|M,\ s \in \mathbb{Z}_p$$

holds. Using the well-known distribution formula of the Bernoulli polynomial

$$M^{k-1} \sum_{l=0}^{M-1} B_k\Big(\frac{l}{M} + t\Big) = B_k(Mt)$$

for a positive integer M and $k \ge 1$, we obtain that

$$\begin{aligned}
\sum_{l=0}^{p-1} B(S, Np^m l + t, Np^{m+1}) &= \sum_{l=0}^{p-1} (Np^{m+1})^{s-1} \omega(Np^m l + t)^{-s} B_s\Big(\frac{Np^m l + t}{Np^{m+1}}\Big) \\
&= (Np^{m+1})^{s-1} \omega(t)^{-s} \sum_{l=0}^{p-1} B_s\Big(\frac{l}{p} + \frac{t}{Np^{m+1}}\Big) \\
&= (Np^m)^{s-1} \omega(t)^{-s} p^{s-1} \sum_{l=0}^{p-1} B_s\Big(\frac{l}{p} + \frac{t}{Np^{m+1}}\Big) \\
&= (Np^m)^{s-1} \omega(t)^{-s} B_s\Big(\frac{t}{Np^m}\Big) \\
&= B(s, t, Np^m).
\end{aligned}$$

Hence

$$\sum_{l=0}^{p-1} B(s, Np^m l + t, Np^{m+1}) = B(s, t, Npt^m) \tag{18}$$

holds and the formula (17) is equal to

$$\sum_{t \equiv a (p^m)} \lambda(t) B(s, t, Np^m) = \mu_{s,\lambda}(a + p^m \mathbb{Z}_p).$$

Therefore we can conclude that $\mu_{s,\lambda}$ is distribution on $\mathbb{Z}_p^\times$. $\square$

Remark 5.1. Let $s = k \ge 0$ be an integer. It can be easily checked that the following is a distribution on $\mathbb{Z}_p$: For $a + p^m \mathbb{Z}_p \subseteq \mathbb{Z}_p$,

$$\mu'_{k,\lambda}(a + p^m \mathbb{Z}_p) = (Np^m)^{k-1} \sum_{\substack{r \equiv a(p^m) \\ 1 \le r < Np^m}} \lambda(r) B_k\Big(\frac{r}{Np^m}\Big). \tag{19}$$

Furthermore we also have $\mu_{k,\lambda} = \mu'_{k,\lambda\omega^{-k}}$. Hence we can extend $\mu_{k,\lambda}$ to $\mathbb{Z}_p$ and from now on we regard $\mu_{k,\lambda}$ as a distribution on $\mathbb{Z}_p$.

Following Mazur's treatment, one can normalize the distribution in order to get a measure on $\mathbb{Z}_p^\times$.

Definition 5.2. *Let α be an integer that is relatively prime to N, $\alpha \equiv 1 \pmod{p}$, and κ be another periodic function of a period M. For an open compact subset K of $\mathbb{Z}_p^\times$, we define a distribution $\mu_{s,\lambda,\kappa,\alpha}$ such that*

$$\mu_{s,\lambda,\kappa,\alpha}(K) = \mu_{s,\lambda}(K) - \alpha^{-s}\mu_{s,\kappa}(\alpha K). \tag{20}$$

We set

$$\mu_{s,\lambda,\alpha} := \mu_{s,\lambda,\lambda,\alpha}.$$

We have the following proposition which is a continuous version of [12], Lemma 7.3.

Theorem 5.4. *(1) $\mu_{s,\lambda,\kappa,\alpha}$ is a measure on $\mathbb{Z}_p^\times$ if and only if*

$$\frac{1}{N}\sum_{r=1}^{N}\lambda(r) = \frac{1}{M}\sum_{r=1}^{M}\kappa(r). \tag{21}$$

In particular, $\mu_{s,\lambda}$ is a measure on $\mathbb{Z}_p^\times$ if and only if $\sum_r \lambda(r) = 0$.
(2) Assuming the conditions (19)*, we obtain that for $s \in \mathbb{Z}_p - \{0\}$*

$$d\mu_{s,\lambda,\kappa,\alpha}(x) = s\langle x\rangle^{s-1} d\mu_{1,\lambda,\kappa,\alpha}(x)$$

and

$$\Big(\lim_{s\to 0} s^{-1} d\mu_{s,\lambda,\kappa,\alpha}\Big)(x) = \langle x\rangle^{-1} d\mu_{1,\lambda,\kappa,\alpha}(x).$$

Proof. From the definition (12) we have

$$\begin{aligned}
\mu_{s,\lambda}(a + p^m\mathbb{Z}_p) &= \sum_r \lambda(r)\frac{\langle r\rangle^r}{Np^m}\sum_{n\geq 0}\binom{s}{n}\Big(\frac{Np^m}{r}\Big)^n B_n &(22)\\
&= \sum_{r\equiv a(p^m}\frac{\langle r\rangle^r}{Np^m}\sum_{n\geq 0}\binom{s}{n}\Big(\frac{Np^m}{r}\Big)^n B_0 + O(1)\\
&= \frac{\langle a\rangle^s B_0}{Np^m}\sum_{n=1}^{N}\lambda(p^m r + a) + O(1), &(23)
\end{aligned}$$

where by $c_n = O(p^{-n\epsilon})$, $\epsilon \geq 0$ we mean that $p^{n\epsilon}c_n$ is p-adically bounded. Let us set $A = N^{-1}\sum_r \lambda(r)$ and $B = M^{-1}\sum_r \kappa(r)$. From the formula of (22) we have the expression

$$\mu_{s,\lambda,\kappa,\alpha}(a + p^m\mathbb{Z}_p) = \frac{A\langle a\rangle^s - B\langle \alpha^{-1}(\alpha a)_m\rangle^s}{p^m}B_0 + O(1),$$

and therefore $\mu_{s,\lambda,\kappa,\alpha}$ is a measure on $\mathbb{Z}_p^\times$ if and only if $A = B$ since $\langle a\rangle^s \equiv \langle \alpha^{-1}(\alpha a)_m\rangle^s \pmod{p^m}$.

From the formula (20), we have

$$\mu_{s,\lambda}(a + p^m\mathbb{Z}_p) = \sum_{r\equiv a(p^m)}\lambda(r)\frac{\langle r\rangle_r}{Np^m} + \lambda(r)s\langle r\rangle^s r^{-1} + O\Big(\frac{1}{p^{m-1}}\Big). \tag{24}$$

We set $r = p^m q + a$ with $0 \leq q < N$. Since $\langle r\rangle^s = \langle a\rangle^s(1 + sp^m qa^{-1}) + O(p^{-2m})$, the formula (23) is equal to

$$\frac{\langle a\rangle^s}{Np^m}\sum_{q=1}^{N}\lambda(q) + s\langle a\rangle^s a^{-1}\sum_{q=1}^{N}\lambda(p^m q + a)\Big(\frac{q}{N} - 1\Big) + O\Big(\frac{1}{p^{m-1}}\Big).$$

If we set

$$\tau(s, \lambda, a) := s\langle a\rangle^s a^{-1}\sum_{q=1}^{N}\lambda(p^m q + a)\Big(\frac{q}{N} - 1\Big),$$

then we have

$$\tau(s, \lambda, a) = s\langle a\rangle^{s-1}\tau(1, \lambda, a).$$

Note that $(a\alpha)_m = \alpha a - p^m g$ where $g = \lfloor \frac{\alpha a}{p^m}\rfloor$ and we have

$$\langle(\alpha a)_m\rangle^s = \langle \alpha a\rangle^s(1 - sp^m g(\alpha a)^{-1}) + O(p^{-2m}).$$

From this, we have

$$\begin{aligned}\mu_{s,\kappa}((\alpha a)_m + p^m\mathbb{Z}_p) &= \frac{\langle \alpha a\rangle^s}{Mp^m}\sum_{q=1}^{M}\kappa(q)\\ &\quad + s\langle \alpha a\rangle^s(\alpha a)^{-1}\sum_{q=1}^{M}\kappa(p^m q + (\alpha a)_m)\Big(\frac{q-g}{M} - 1\Big) + O\Big(\frac{1}{p^{m-1}}\Big).\end{aligned}$$

Similarly we also set

$$\tau'(s, \kappa, a) := s\langle a\rangle^s a^{-1}\sum_{q=1}^{M}\kappa(p^m q + (\alpha a)_m)\Big(\frac{q-g}{M} - 1\Big),$$

and we have

$$\tau'(s, \kappa, a) = s\langle a\rangle^{s-1}\tau'(1, \kappa, a).$$

In total, we have

$$\begin{aligned}\mu_{s,\lambda,\kappa,\alpha}(a + p^m\mathbb{Z}_p) &= \mu_{s,\lambda}(a + p^m\mathbb{Z}_p) - \langle\alpha\rangle^{-s}\mu_{s,\lambda}((\alpha a)_m + p^m\mathbb{Z}_p)\\ &= \tau(s, \lambda, a) - \alpha^{-1}\tau'(s, \kappa, (\alpha a)_m) + O\Big(\frac{1}{p^{m-1}}\Big)\\ &= s\langle a\rangle^{s-1}(\tau(1, \lambda, a) - \alpha^{-1}\tau'(1, \kappa, a)) + O\Big(\frac{1}{p^{m-1}}\Big)\\ &= s\langle a\rangle^{s-1}\mu_{1,\lambda,\kappa,\alpha}(a + p^m\mathbb{Z}_p) + O\Big(\frac{1}{p^{m-1}}\Big).\end{aligned}$$

To finish the proof, let us consider $|\langle x\rangle^{s-1} - \langle x\rangle^{-1}|_p$ for $x \in \mathbb{Z}_p^\times$. We have

$$|\langle x\rangle^{s-1} - \langle x\rangle^{-1}|_p = |\langle x\rangle^s - 1|_p \leq |s|_p|\langle x\rangle - 1|_p.$$

Hence for all function f on $\mathbb{Z}_p^\times$ with $\|f\|_p \le 1$, we have

$$\begin{aligned}\int_U f(x)d\mu_{s,\lambda,\kappa,\alpha}(x) &= \lim\Big(\sum_{a+p^m\mathbb{Z}_p\subseteq U} f(a)\mu_{s,\lambda,\kappa,\alpha}(a+p^m\mathbb{Z}_p)\Big)\\ &\equiv \lim \sum_{a+p^m\mathbb{Z}_p\subseteq U} f(a)s\langle a\rangle^{s-1}\mu_{1,\lambda,\kappa,\alpha}(a+p^m\mathbb{Z}_p) \pmod{p^m}\\ &= \int_U f(x)s\langle x\rangle^{s-1}\mu_{1,\lambda,\kappa,\alpha}(x).\end{aligned}$$

Since the above inequality is independent of f, we obtain the result. □

Let Ψ be a periodic function with a period cN for a p-power c and $p \nmid N$. Then Ψ can be written as $\Psi = \sum_i \psi_i\lambda_i$ where ψ_i and λ_i are periodic functions with periods c and N, respectively.

Definition 5.3. *For $\alpha \equiv 1 \pmod{cp}$, we set*

$$d\sigma_{\Psi,\alpha}(x) = \sum_{i=1}^m \psi_i(x)d\mu_{1,\lambda_i,\alpha}(x).$$

Note that $d\sigma_{\Psi,\alpha}(x) = \psi(x)d\mu_{1,\lambda,\alpha}(x)$ if and only if

$$\int_{\mathbb{Z}_p^\times} f(x)d\sigma_{\Psi,\alpha}(x) = \int_{\mathbb{Z}_p^\times} f(x)\psi(x)d\mu_{1,\lambda,\alpha}(x)$$

for all $f \in C(\mathbb{Z}_p^\times, \mathbb{C}_p)$.

Corollary 5.5. *Let $\alpha \equiv 1 \pmod{cp}$. Then we have*

$$\int_{\mathbb{Z}_p^\times} \langle x\rangle^{-1}d\sigma_{\Psi,\alpha}(x) = \frac{\log_p(\alpha)}{cN}\sum_{\substack{r=1\\ p\nmid r}}^{cN}\Psi(r).$$

Proof. First let us consider the case that $\Psi = \psi\lambda$ for $\psi \in \mathcal{L}(\mathbb{Z}_p, \overline{\mathbb{Z}}_p)$ and $\lambda \in \mathcal{L}(\mathbb{Z}/n\mathbb{Z}, \overline{\mathbb{Z}}_p)$. Let c be a period of ψ. Then we have

$$d\sigma_{\Psi,\alpha}(x) = \psi(x)d\mu_{1,\lambda,\alpha}(x).$$

From Theorem 5.4, we obtain

$$\int_{\mathbb{Z}_p^\times} \langle x\rangle^{-1}d\sigma_{\Psi,\alpha}(x) = \lim_{s\to 0}\int_{\mathbb{Z}_p^\times} s^{-1}\psi(x)d\mu_{s,\lambda,\alpha}(x). \tag{25}$$

Since ψ is a locally constant function, we have

$$\begin{aligned}\int_{a+c\mathbb{Z}_p}\psi(x)d\mu_{s,\lambda,\alpha}(x) &= \psi(a)\int_{a+c\mathbb{Z}_p} 1d\mu_{s,\lambda,\alpha}(x)\\ &= \psi(a)\mu_{s,\lambda,\alpha}(a+c\mathbb{Z}_p).\end{aligned}$$

Also since $\mathbb{Z}_p^\times = \bigcup_{1\le a\le c-1//(a,c)=1} (a+c\mathbb{Z}_p)$, the formula (25) is equal to

$$\begin{aligned}\sum_{\alpha\in(\mathbb{Z}/c\mathbb{Z})^\times} \psi(a)\lim_{s\to 0} s^{-1}(\mu_{s,\lambda}(a+c\mathbb{Z}_p) - \alpha^{-s}\mu_{s,\lambda}(\alpha a+c\mathbb{Z}_p))\\ &= \lim_{s\to 0} s^{-1}(1-\alpha^{-s})\sum_a \psi(a)\mu_{0,\lambda}(a+c\mathbb{Z}_p)\\ &= \log_p(\alpha)\sum_a \psi(a)\sum_{r\equiv a(c)} \lambda(r)\\ &= \frac{\log_p(\alpha)}{cN}\sum_{\substack{r=1\\ p\nmid r}}^{cN} \psi\lambda(r).\end{aligned}$$

For a general $\Psi : \mathbb{Z}/cN\mathbb{Z} \to$A, Ψ is a linear combination of $\psi\lambda$, i.e.,$\Psi = \sum_{i,j} c_{ij}\psi_i\lambda_j$ for $c_{ij} \in \mathbb{C}_p$. Thus by the linearity, we extend the above calculations to a general Ψ. The corollary is verified. □

Definition 5.4. *For $s \in \mathbb{Z}_p$, we define two functions B_s and D_s on the basic open sets of $1+p\mathbb{Z}_p$ such that*

$$B_s(a+p^m\mathbb{Z}_p) := B(s,a,p^m) \text{ and } D_s := \frac{d}{ds}B_s.$$

Observe that the relation (18) can be rewritten as

$$\sum_{l=0}^{q-1} B(s, a+Ml, Mq) = B(s,a,M) \text{ with } p|M,\ s\in\mathbb{Z}_p.$$

It follows that B_s and D_s are distributions on $1+p\mathbb{Z}_p$. We also normalize the distributions as follows: For $\alpha \equiv 1 \pmod p$ and an open set K of $1+p\mathbb{Z}_p$,

$$B_{s,\alpha}(K) = B_s(K) - \alpha^{-s}B_s(\alpha K) \text{ and } D_{s,\alpha} = \frac{d}{ds}B_{s,\alpha}.$$

The distribution $B_{s,\alpha}$ becomes a measure on $1+p\mathbb{Z}_p$ since we have the expression

$$B_{s,\alpha}(a+p^m\mathbb{Z}_p) = \frac{a^s - (\alpha^{-1}(\alpha a)_m)^s}{p^m}B_0 + \sum_{r=1}^{\infty}\binom{s}{r}p^{m(r-1)}(a^{s-r} - \alpha^{-s}(\alpha a)_m^{s-r})B_r,$$

which is p-adically bounded. $D_{s,\alpha}$ is also a measure since the terms $\frac{d}{ds}\binom{s}{r}p^{m(r-1)}$, $r \ge 0$ are bounded. We have

$$\frac{d}{dz}\binom{z}{k} = \sum_{i=0}^{k-1}\frac{\binom{z}{i}\binom{z-i-1}{k-i-1}}{k\binom{k-1}{i}} = \binom{z}{k}\sum_{i=0}^{k-1}\frac{1}{z-i}.$$

Then

$$\begin{aligned}\frac{d}{ds}\binom{s}{r}p^{m(r-1)} &= p^{m(r-1)}\frac{d}{ds}\binom{s}{r}\\ &= p^{m(r-1)}\binom{s}{r}\sum_{i=0}^{r-1}\frac{1}{r-i}\\ &= p^{m(r-1)}\binom{s}{r}\sum_{i=1}^{r}\frac{1}{i}.\end{aligned}$$

Since $1 \le i \le r$, we have $v_p(i) < r$. Then $|i|_p > p^{-r}$, and it implies that $\frac{1}{|i|_p} \le p^r$. So, we get $|\sum_{i=1}^{r} \frac{1}{i}|_p \le \max_i(\frac{1}{|i|_p}) \le p^r$. Since $|\binom{s}{r}|_p \le 1$, we have $|p^{m(r-1)}\binom{s}{r}\sum_{i=0}^{r-1}\frac{1}{r-i}|_p \le p^{-m(r-1)}p^r \le 1$ for $m \ge 2$. Hence we conclude that $\frac{d}{ds}\binom{s}{r}p^{m(r-1)}$, $r \ge 0$ are bounded, and it implies that $D_{s,\alpha}$ is also a measure.

Corollary 5.6.

$$\int_{1+p\mathbb{Z}_p} x^{-1}\log_p(x)dB_{1,\alpha}(x) = \log_p(\alpha)(D_0(1+p\mathbb{Z}_p) - \frac{\log_p \alpha}{2p}).$$

Proof. From Theorem 5.4, we have

$$dD_{s,\alpha}(x) = x^{s-1}B_{1,\alpha}(x) + sx^{s-1}\log_p(x)dB_{1,\alpha}(x).$$

This enables us to have the calculations

$$\begin{aligned}\int_{1+p\mathbb{Z}_p} x^{-1}\log_p(x)dB_{1,\alpha}(x) &= \lim_{s\to 0}\int_{1+p\mathbb{Z}_p} x^{s-1}\log_p(x)dB_{1,\alpha}(x)\\ &= \lim_{s\to 0}\frac{1}{s}\Big\{\int_{1+p\mathbb{Z}_p} dD_{s,\alpha} - \int_{1+p\mathbb{Z}_p} x^{s-1}dB_{1,\alpha}(x)\Big\}. \qquad (26)\end{aligned}$$

Let $B_s \circ \alpha(a+p^m\mathbb{Z}_p) := B_s(\alpha a + p^m\mathbb{Z}_p)$ and $dB_s \circ \alpha(x) := dB_s(\alpha x)$. The definition of $B_{s,\alpha}$ gives us $D_{s,\alpha} = D_s + \alpha^{-s}\log_p(\alpha)\circ\alpha - \alpha^{-s}D_s\circ\alpha$. Since $\alpha \equiv 1(modp)$, we have $\alpha(1+p\mathbb{Z}_p) = 1+p\mathbb{Z}_p$. Then we obtain that

$$\begin{aligned}\int_{1+p\mathbb{Z}_p} dD_{s,\alpha} &= D_{s,\alpha}(1+p\mathbb{Z}_p)\\ &= D_s(1+p\mathbb{Z}_p) + \alpha^{-s}\log_p(\alpha)B_s(\alpha(1+p\mathbb{Z}_p)) - \alpha^{-s}D_s(\alpha(1+p\mathbb{Z}_p))\\ &= (1-\alpha^{-s})D_s(1+p\mathbb{Z}_p) + \frac{\log_p(\alpha)}{\alpha^s}B_s(1+p\mathbb{Z}_p).\end{aligned}$$

From this, the formula (26) is equal to

$$\begin{aligned}&\lim_{s\to 0}\frac{1}{s}\Big\{(1-\alpha^{-s})D_s(1+p\mathbb{Z}_p) + \frac{\log_p(\alpha)}{\alpha^s}B_s(1+p\mathbb{Z}_p) - \int_{1+p\mathbb{Z}_p} x^{s-1}dB_{1,\alpha}(x)\Big\}\\ &= \log_p(\alpha)D_0(1+p\mathbb{Z}_p) + \lim_{s\to 0}\frac{1}{s}\Big\{\frac{\log_p(\alpha)}{\alpha^s}B_s(1+p\mathbb{Z}_p) - \int_{1+p\mathbb{Z}_p} x^{s-1}dB_{1,\alpha}(s)\Big\}.\end{aligned}$$

The second term in the last formula is equal to

$$-(\log_p(\alpha))^2B_0(1+p\mathbb{Z}_p) + \log_p(\alpha)D_0(1+p\mathbb{Z}_p) - \int_{1+p\mathbb{Z}_p} x^{-1}\log_p(x)dB_{1,\alpha}(x).$$

In total, we have

$$\begin{aligned}\int_{1+p\mathbb{Z}_p} x^{-1}\log_p(x)dB_{1,\alpha}(x) =& \log_p(\alpha)D_0(1+p\mathbb{Z}_p) - (\log_p(\alpha))^2B_0(1+p\mathbb{Z}_p)\\ &+ \log_p(\alpha)D_0(1+p\mathbb{Z}_p) - \int_{1+p\mathbb{Z}_p} x^{-1}\log_p(x)dB_{1,\alpha}(x).\end{aligned}$$

Then we obtain

$$2\int_{1+p\mathbb{Z}_p} x^{-1}\log_p(x)dB_{1,\alpha}(x) = 2\log_p(\alpha)D_0(1+p\mathbb{Z}_p) - (\log_p(\alpha))^2B_0(1+p\mathbb{Z}_p).$$

By Definition 5.4 and the equation (11), we have $B_0(1+p\mathbb{Z}_p) = B(0,1,p) = \frac{1}{p}$. The corollary is verified. □

6 p-adic periodic zeta function

Let Ψ be a periodic function with a period M.

Definition 6.1. *The periodic zeta function $L(s,\Psi)$ is defined by*

$$L(s,\Psi)=\sum_{n\geq 1}\frac{\Psi(n)}{n^s} \text{ for } \Re(s)>1.$$

The function $L(s,\Psi)$ has the meromorphic continuation to $\mathbb{C}$ with a simple pole at $s=1$ with the residue $\frac{1}{M}\sum_{r=1}^{M}\Psi(r)$. In fact, $L(s,\Psi)$ can be represented by the integral

$$\Gamma(s)L(s,\Psi)=\int_0^\infty R_\Psi(e^{-y})y^{s-1}dy \text{ for } \Re(s)>1 \tag{27}$$

and

$$(e^{2\pi is}-1)\Gamma(s)L(s,\Psi)=\int_{P(\rho)}R_\Psi(e^{-y})y^{s-1}dy \text{ for } s\in\mathbb{C},$$

where the branch of log, $0\leq log(y)<2\pi$ is used for y^s, $P(\rho)$ is a positively oriented contour $\{\rho e^{i\theta}|0\leq\theta<2\pi\}\cup\{te^{i\theta}|\rho<t<\infty,\theta=0,2\pi\}$ with sufficiently small $\rho>0$, and

$$R_\Psi(q):=\frac{\sum_{r=1}^{M}\Psi(r)q^r}{1-q^M}.$$

We also have the expression

$$L(s,\Psi)=N^{-s}\sum_r\Psi(r)\zeta(s,\frac{r}{N}) \tag{28}$$

which enables us to get the functional equation (See [10]) of $L(s,\Psi)$ as follows:

$$(e^{2\pi is}-1)\Gamma(s)L(s,\Psi)=\Big(\frac{2\pi i}{M}\Big)^s\Big(L(1-s,\hat{\Psi})-(-1)^{s-1}L(1-s,(-1)^*\hat{\Psi})\Big),$$

where $\hat{\Psi}$ is the Fourier transform of Ψ defined by

$$\hat{\Psi}(r)=\sum_{t=0}^{M-1}\Psi(t)\zeta_M^{-rt}.$$

From (28), we obtain

$$L(1-k,\Psi)=-\frac{M^{k-1}}{k}\sum_r\Psi(r)B_k\Big(\frac{r}{M}\Big). \tag{29}$$

Now, let $M=cN$ for a p-power c and a positive integer N with $p\nmid N$. We define $\Psi^{(p)}$ as

$$\Psi^{(p)}=\begin{cases}\Psi(r) & \text{if } p\nmid r\\ 0 & \text{if } p|r\end{cases}.$$

Theorem 6.1. *There exists a p-adic period zeta function $L_p(s,\Psi)$ which is a p-adic meromorphic function on $\mathbb{Z}_p$ with a simple pole at $s=1$ and the residue*

$$Res_{s=1}L_p(s,\Psi)=\frac{1}{cN}\sum_{\substack{r=1\\ p\nmid r}}^{cN}\Psi(r). \tag{30}$$

It has the interpolation property as follows:

$$L_p(1-k,\Psi) = L(1-k,\Psi_k^{(p)}) \textit{ for all } k \geq 1,$$

where $\Psi_k = \Psi\omega^{-k}$.

Proof. Choose an integer α so that $\alpha \equiv 1 (\bmod\, cp)$. Define the p-adic periodic zeta function of Ψ by

$$L_p(s,\Psi) = \frac{1}{\alpha^{s-1}-1}\int_{\mathbb{Z}_p^\times} \langle x\rangle^{-s} d\sigma_{\Psi,\alpha}(x).$$

Note that it is a p-adic meromorphic function on $\mathbb{Z}_p$ with the pole at $s=1$ and the residue

$$Res_{s=1}L_p(s,\Psi) = \frac{1}{\log_p \alpha}\int_{\mathbb{Z}_p^\times} \langle x\rangle^{-1} d\sigma_{\Psi,\alpha} = \frac{1}{cN}\sum_{\substack{r=1\\ p\nmid N}}^{cN} \Psi(r).$$

Suppose first that $\Psi = \psi\lambda$. We have the calculations

$$\int_{\mathbb{Z}_p^\times} \langle x\rangle^{k-1} d\sigma_{\Psi,\alpha} = \frac{1}{k}\int_{\mathbb{Z}_p^\times} \psi(x) d\mu_{k,\lambda,\alpha} = \frac{1}{k}\sum_{\substack{x=1\\ p\nmid x}}^{c} \psi(x)\mu_{k,\lambda,\alpha}(x+c\mathbb{Z}_p).$$

Note that from (12) and (29) we have

$$\begin{aligned}\sum_{p\nmid x}\psi(x)\mu_{k,\lambda}(x+c\mathbb{Z}_p) &= (cN)^{k-1}\sum_{p\nmid x}\psi(x)\sum_{r\equiv x(c)}\lambda(r)\omega^{-k}(r)B_k\Big(\frac{r}{cN}\Big)\\ &= -kL(1-k,(\psi\lambda\omega^{-k})^{(p)}).\end{aligned}$$

Hence we have the interpolation.

$$\int_{\mathbb{Z}_p^\times} \langle x\rangle^{k-1} d\sigma_{\Psi,\alpha} = (\alpha^{-k}-1)L(1-k,(\psi\lambda\omega^{-k})^{(p)}). \tag{31}$$

By extending (31) linearly to a general Ψ, we conclude the theorem. □

The following construction of p-adic Dirichlet L-function is well-known:

Corollary 6.2. *Let* $\chi : (\mathbb{Z}/cN\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ *be a Dirichlet character with the conductor* cN *where* c *is a* p*-power and* $(p,N)=1$*. There exists a* p*-adic analytic function* $L_p(s,\chi)$ *unless* $\chi = 1$ *so that*

$$L_p(1-k,\chi) = L(1-k,\chi_k^{(p)}) = (1-\chi\omega^{-k}(p)p^{k-1})L(1-k,\chi\omega^{-k}).$$

If $\chi = 1$*, then* $L_p(s,1)$ *is a* p*-adic meromorphic function on* $\mathbb{Z}_p$ *with the simple pole at* $s=1$ *and its residue is* $1-p^{-1}$.

Proof. The residue of $L_p(s,\chi)$ at $s=1$ is

$$\frac{1}{cN}\sum_{\substack{r=1\\ (Np,r)=1}}^{cN-1}\chi(r) = \begin{cases} 0 & \text{if } \chi \neq 1\\ \frac{p-1}{p} & \text{otherwise}\end{cases}$$

This concludes the proof. □

Remark 6.1. Observe that if $\chi(-1) = -1$ then $L_p(s, \chi)$ is identically zero. Indeed, if we set $\chi = \chi_p \chi_N$ such that the conductors of χ_p and χ_N are a p-power and N respectively, then we have the calculations

$$\mu_{1,\chi_N}(-x + p^m \mathbb{Z}_p) = \sum_{\substack{1 \le r < cN \\ r \equiv x(c)}} \chi_N \omega(-r) B_1\Big(1 - \frac{r}{cN}\Big).$$

Since we have $B_1(1 - x) = -B_1(x)$, we have

$$d\mu_{1,\chi_N,\alpha} \circ -1 = -\chi_N \omega(-1) d\mu_{1,\chi_N,\alpha}.$$

In conclusion, we obtain

$$\int_{\mathbb{Z}_p^\times} \chi_p(x) \langle x \rangle^{-s} d\mu_{1,\chi_N,\alpha}(x) = \chi(-1) \int_{\mathbb{Z}_p^\times} \chi_p(x) \langle x \rangle^{-s} d\mu_{1,\chi_N,\alpha}(x).$$

Therefore the p-adic L-function is identically zero if $\chi(-1) = -1$.

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